

# EULER-MAHONIAN DISTRIBUTIONS OF TYPE $B_n$

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**ABSTRACT.** Adin, Brenti, and Roichman introduced the pairs of statistics  $(\text{ndes}, \text{nmaj})$  and  $(\text{fdes}, \text{fmaj})$ . They showed that these pairs are equidistributed over the hyperoctahedral group  $B_n$ , and can be considered “Euler-Mahonian” in that they generalize the Carlitz identity. Further, they asked whether there exists a bijective proof of the equidistribution of their statistics. We give such a bijection, along with a new proof of the generalized Carlitz identity.

## 1. INTRODUCTION

Statistics such as inversion number, major index, and descent number have been well studied for the symmetric group  $S_n$ . MacMahon [7] showed that inversion number and major index are equidistributed and Carlitz [4] described the joint distribution of descent and major index—the “Euler-Mahonian” distribution—with the following theorem.

**Theorem 1** ([4]). *For positive integers  $n$ ,*

$$(1) \quad \sum_{r \geq 0} [r+1]_q^n t^r = \frac{\sum_{u \in S_n} t^{\text{des}(u)} q^{\text{maj}(u)}}{\prod_{i=0}^n (1 - tq^i)},$$

where  $[r+1]_q = 1 + q + \cdots + q^r$ .

The Coxeter group generalization of inversion number is *length*. (An element  $w$  in a Coxeter group has length  $k$  if  $w = s_1 \cdots s_k$  is a minimal expression for  $w$  as a product of simple reflections.) Adin and Roichman [3] generalized MacMahon’s result to the hyperoctahedral group  $B_n$  in demonstrating that a new statistic, the *flag major* statistic, is equidistributed with length. Adin, Brenti, and Roichman [1], introduced new statistics on  $B_n$ , the *negative descent*, the *negative major*, and the *flag descent*. They proved that the pairs of statistics  $(\text{fdes}, \text{fmaj})$  and  $(\text{ndes}, \text{nmaj})$  are equidistributed over  $B_n$ . Moreover, they showed these bivariate distributions are “type  $B_n$  Euler-Mahonian” in the sense that they generalize Theorem 1 as follows. (Chow and Gessel provide an alternative generalization in [5].)

**Theorem 2.** *For positive integers  $n$ ,*

$$(2) \quad \sum_{r \geq 0} [r+1]_q^n t^r = \frac{\sum_{w \in B_n} t^{\text{ndes}(w)} q^{\text{nmaj}(w)}}{(1-t) \prod_{i=1}^n (1 - t^2 q^{2i})} \quad [1, \text{Theorem 3.2}],$$

$$(3) \quad = \frac{\sum_{w \in B_n} t^{\text{fdes}(w)} q^{\text{fmaj}(w)}}{(1-t) \prod_{i=1}^n (1 - t^2 q^{2i})} \quad [1, \text{Theorem 4.2}].$$

One of the drawbacks of [1] is that while their proof of (2) is quite brief and elementary, their proof of (3) is rather indirect and tedious. Further, they remark (see the discussion after [1, Corollary 4.5]) that “it would be interesting to have a direct combinatorial (i.e., bijective)

proof” that these type  $B_n$  Euler-Mahonian statistics are equidistributed. The purpose of this note is to provide such a proof (Theorem 7). Along the way we give new combinatorial proofs of (2) and (3) (Theorems 3 and 5, respectively).

## 2. OVERVIEW

To any word  $w = w_1 \cdots w_n$  whose letters are totally ordered, let  $\text{Des}(w) := \{i : w_i > w_{i+1}\}$ , and let  $\text{des}(w) := |\text{Des}(w)|$ , called the *descent set* and *descent number*, respectively. We define the *major statistic* as

$$\text{maj}(w) := \sum_{i \in \text{Des}(w)} i.$$

Let  $S_n$  denote the set of all permutations of the set  $[n] := \{1, 2, \dots, n\}$ . The *refined Eulerian polynomial* is the generating function for the joint distribution of  $(\text{des}, \text{maj})$  over  $S_n$ :

$$S_n(t, q) := \sum_{u \in S_n} t^{\text{des}(u)} q^{\text{maj}(u)}.$$

This function is the numerator of the right-hand side of (1). For  $q = 1$ , we have  $S_n(t) = \sum_{u \in S_n} t^{\text{des}(u)}$ , the classical Eulerian polynomial shifted by  $t^{-1}$ .

The hyperoctahedral group,  $B_n$ , has as its elements all *signed permutations* of  $[n]$ . For our purposes, these are all words  $w = w_1 \cdots w_n$  on the alphabet

$$[n] \cup [\bar{n}] = \{\bar{1}, 1, \bar{2}, 2, \dots, \bar{n}, n\}$$

such that  $|w| = |w_1| \cdots |w_n|$  is a permutation in  $S_n$ . (We write  $\bar{i}$  for  $-i$ .) The precise way in which the flag and negative statistics are defined for  $w$  in  $B_n$  depends on how we choose to put a total order on  $[n] \cup [\bar{n}]$ , though the distribution over  $B_n$  is independent of this choice. It will be most convenient to have the lexicographic order:

$$\bar{1} < \bar{2} < \cdots < \bar{n} < 1 < 2 < \cdots < n.$$

We remark that [3] uses this order, whereas [1] uses the usual integer ordering.

Our approach is to provide two very explicit combinatorial proofs that the pairs of statistics in question are distributed over  $B_n$  as

$$S_n(t, q) \cdot \prod_{i=1}^n (1 + tq^i).$$

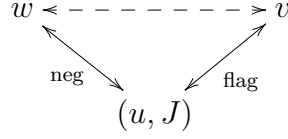
For  $q = 1$  this is the intriguing formula  $(1+t)^n S_n(t)$ . (The first author in fact has a different combinatorial proof that flag descents are distributed in this fashion; see [6].)

In fact, we will demonstrate a finer result. Namely, that for each pair of statistics we have a way of assigning  $2^n$  signed permutations to each unsigned permutation  $u \in S_n$  so that the resulting collection has distribution

$$t^{\text{des}(u)} q^{\text{maj}(u)} \prod_{i=1}^n (1 + tq^i).$$

The signed permutations in this collection are thus identified with a pair  $(u, J)$ ,  $J \subset [n]$ , with “weight”  $t^{\text{des}(u)+|J|} q^{\text{maj}(u)+\sum_{j \in J} j}$ . This weight-preserving bijection can then be composed to

obtain the desired bijection. If  $w$  corresponds to  $(u, J)$  with respect to  $(\text{ndes}, \text{nmaj})$  and  $v$  corresponds to  $(u, J)$  with respect to  $(\text{fdes}, \text{fmaj})$ , then we identify  $w$  and  $v$  with one another:



This idea is formalized with Theorem 7.

The paper is organized as follows. Section 3 presents the negative statistics, Section 4 discusses the flag statistics, and Section 5 exhibits the desired bijection.

### 3. THE NEGATIVE STATISTICS

As in [1], define the *negative descent number* of  $w \in B_n$  as

$$\text{ndes}(w) := \text{des}(w) + |\{i : w_i < 0\}|,$$

and the *negative major index* as

$$\text{nmaj}(w) := \text{maj}(w) + \sum_{w_i < 0} |w_i|.$$

For example if  $w = \bar{5}7612\bar{4}\bar{3}$ ,  $\text{ndes}(w) = 4 + 3 = 7$ , and  $\text{nmaj}(w) = 16 + 12 = 28$ . Let

$$B_n^{(\text{neg})}(t, q) := \sum_{w \in B_n} t^{\text{ndes}(w)} q^{\text{nmaj}(w)},$$

be the generating function for this pair of statistics.

To any  $u = u_1 \cdots u_n$  in  $S_n$  we can associate  $2^n$  signed permutations in the following manner. Define the *standardization* of a signed permutation  $w \in B_n$ ,  $\text{st}(w)$ , to be the unsigned permutation in  $S_n$  that is obtained by replacing the smallest letter of  $w$  with 1, the next smallest with 2, and so on. For example,  $\text{st}(\bar{5}7612\bar{4}\bar{3}) = 3764521$ .

Now given any  $u$  in  $S_n$ , let

$$B(u) = \{w \in B_n : \text{st}(w) = u\}.$$

We have

$$B_n = \bigcup_{u \in S_n} B(u) \quad (\text{disjoint union}).$$

**Remark.** This partition of  $B_n$  is employed by Adin, Brenti, and Roichman (see equation (5) of [1]), though in different language. It plays a role in their proof of their Theorem 3.2 (equation (2) above) and a refined version of the same result from a different work [2, Theorem 6.7]. This partition will play a role in our proof as well, though the proofs are fundamentally distinct.

It is easy to see that every element of  $B(u)$  is uniquely determined by its set of negative letters. Let  $u_J$  denote the member of  $B(u)$  that is a permutation of the set

$$\{i : i \in [n] \setminus J\} \cup \{\bar{j} : j \in J\}.$$

For example, if  $u = 3142$ , then  $u_{\{1,3\}} = 2\bar{1}4\bar{3}$  since  $\bar{1}$  and  $\bar{3}$  are the negative letters and  $\text{st}(2\bar{1}4\bar{3}) = 3142$ .

The following result and Theorem 1 implies equation (2) of Theorem 2.

**Theorem 3.** *We have*

$$B_n^{(\text{neg})}(t, q) = S_n(t, q) \cdot \prod_{i=1}^n (1 + tq^i).$$

Theorem 3 follows immediately from the subsequent stronger claim.

**Theorem 4.** *For any  $u \in S_n$ , we have*

$$t^{\text{des}(u)} q^{\text{maj}(u)} \prod_{i=1}^n (1 + tq^i) = \sum_{w \in B(u)} t^{\text{ndes}(w)} q^{\text{nmaj}(w)}.$$

*Proof.* Let  $w = u_J$  for any  $J \subset [n]$ . We want to show that

$$t^{\text{ndes}(w)} q^{\text{nmaj}(w)} = t^{\text{des}(u)} t^{\text{maj}(u)} \prod_{j \in J} tq^j.$$

Since  $\text{st}(w) = u$ , we have  $\text{Des}(w) = \text{Des}(u)$ . In particular,  $\text{des}(w) = \text{des}(u)$  and  $\text{maj}(w) = \text{maj}(u)$ . By definition of  $w$ , we know  $\{w_i : w_i < 0\} = J$ . Comparing with the definition of the negative statistics, this completes the proof.  $\square$

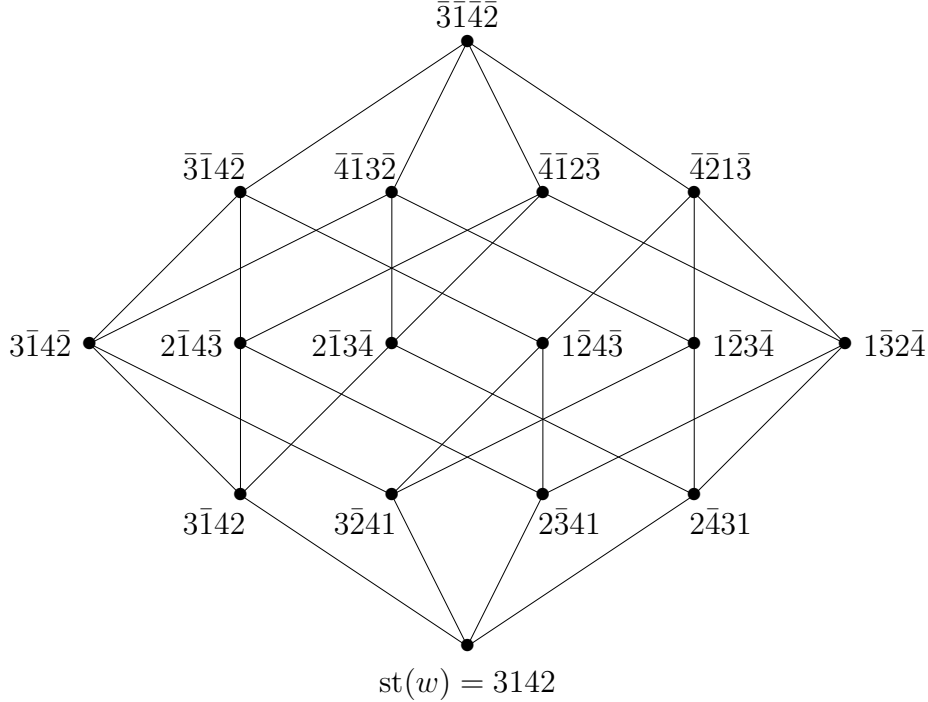


FIGURE 1.  $B(u)$  for  $u = 3142$

It is informative to observe that Theorem 4 implies the set  $B(u)$  has, up to a shift, a weight-preserving bijection with a boolean algebra given by

$$J \longleftrightarrow u_J.$$

That is, if  $w \in B(u)$  corresponds to  $J = \{j_1, \dots, j_k\}$ , then  $\text{ndes}(w) = \text{des}(u) + k$  and  $\text{nmaj}(w) = \text{maj}(u) + (j_1 + \dots + j_k)$ . Figure 1 provides an illustration for  $u = 3142$ .

## 4. THE FLAG STATISTICS

From [1], our definitions for the flag statistics are as follows. For  $w \in B_n$ ,

$$\text{fdes}(w) := \begin{cases} 2 \text{des}(w) + 1 & \text{if } w_1 < 0 \\ 2 \text{des}(w) & \text{if } w_1 > 0, \end{cases}$$

and

$$\text{fmaj}(w) := 2 \text{maj}(w) + |\{i : w_i < 0\}|.$$

So, for example, if  $w = \bar{5}7612\bar{4}\bar{3}$ , then  $\text{Des}(w) = \{2, 3, 5, 6\}$ ,  $\text{fdes}(w) = 2 \cdot 4 + 1 = 9$ , and  $\text{fmaj}(w) = 2 \cdot 16 + 3 = 35$ . Let  $\text{wt}(w) = t^{\text{fdes}(w)} q^{\text{fmaj}(w)}$  be the *weight* of  $w$ . The generating function is:

$$B_n^{(\text{flag})}(t, q) = \sum_{w \in B_n} \text{wt}(w) = \sum_{w \in B_n} t^{\text{fdes}(w)} q^{\text{fmaj}(w)}.$$

Before proceeding, it will be helpful to provide another partition of  $B_n$ . To any  $u = u_1 \cdots u_n$  in  $S_n$ , we can assign  $2^n$  signed permutations by independently assigning minus signs to the letters of  $u$  in all possible ways. That is, let

$$B'(u) = \{w \in B_n : |w| = u\}.$$

We have

$$B_n = \bigcup_{u \in S_n} B'(u) \quad (\text{disjoint union}).$$

Just as with Theorem 3, we have the following, which, with Theorem 1 completes the proof of Theorem 2.

**Theorem 5** ([1], Theorem 4.4). *We have*

$$B_n^{(\text{flag})}(t, q) = S_n(t, q) \cdot \prod_{i=1}^n (1 + tq^i).$$

**Corollary 1** ([1], Corollary 4.5). *The statistics  $(\text{fdes}, \text{fmaj})$  and  $(\text{ndes}, \text{nmaj})$  are equidistributed, i.e.,*

$$B_n^{(\text{flag})}(t, q) = B_n^{(\text{neg})}(t, q).$$

Theorem 5 will follow from the following theorem, analogous to Theorem 3. With the combinatorial proof of Theorem 6 in hand we will be able to provide the desired bijective proof of Corollary 1.

**Theorem 6.** *For any  $u$  in  $S_n$ , we have:*

$$t^{\text{des}(u)} q^{\text{maj}(u)} \prod_{i=1}^n (1 + tq^i) = \sum_{w \in B'(u)} t^{\text{fdes}(w)} q^{\text{fmaj}(w)}.$$

Theorem 6 will follow from Lemma 2. First, we need one more idea. Let  $\Delta_i$  be the operator that negates the first  $i$  letters of a signed permutation, i.e.,

$$\Delta_i w = \overline{w_1 \cdots w_i} w_{i+1} \cdots w_n.$$

**Lemma 1.** *Fix  $u \in S_n$  and let  $w \in B'(u)$  be such that all the letters  $w_1, \dots, w_{j+1}$  have the same sign. Then*

$$\text{wt}(\Delta_j w) = \begin{cases} \text{wt}(w)/tq^j & \text{if } j \in \text{Des}(u) \\ \text{wt}(w) \cdot tq^j & \text{if } j \notin \text{Des}(u). \end{cases}$$

*Proof.* Suppose everything to the left of  $w_{j+1}$  (inclusive) is positive. Then if  $j \notin \text{Des}(u)$  we have lost nothing and gained  $tq^j$ , since  $w_1$  is now negative and we have  $j$  new negative numbers. If we have  $j \in \text{Des}(u)$ , then we still gain  $tq^j$ , but we have lost the descent in  $j$ , and with it  $t^2q^{2j}$ , for a net weight change of  $t^{-1}q^{-j}$ .

If everything to the left of  $w_{j+1}$  is negative the situation is similar. If  $j \in \text{Des}(u)$  we have gained no new descents, lost  $j$  negative numbers, and changed  $w_1$  from negative to positive; a net weight change of  $t^{-1}q^{-j}$ . If we have  $j \notin \text{Des}(u)$ , then we gain  $t^2q^{2j}$  as  $j$  becomes a descent, but we have lost  $j$  negative numbers and  $w_1$  has gone from negative to positive, yielding a change of  $t^2q^{2j}/tq^j = tq^j$ .

This completes the proof. □

Notice that the  $\Delta_i$  commute, so for a subset  $J = \{j_1 < \dots < j_k\}$  of  $[n]$ , there is no ambiguity in defining the composite operator  $\Delta_J = \Delta_{j_1} \cdots \Delta_{j_k}$ .

**Lemma 2.** *For any  $u \in S_n$ , we have*

$$\text{wt}(\Delta_J u) = t^{\text{des}(u)} q^{\text{maj}(u)} \prod_{j \in J \Delta \text{Des}(u)} tq^j,$$

where  $J \Delta \text{Des}(u) = (J \cup \text{Des}(u)) \setminus (J \cap \text{Des}(u))$  is the symmetric difference of  $J$  and  $\text{Des}(u)$ .

*Proof.* First notice that  $\text{fdes}(u) = 2 \text{des}(u)$  and  $\text{fmaj}(u) = 2 \text{maj}(u)$ . Since the  $\Delta_j$  commute, we can apply  $\Delta_{j_k}$  first,  $\Delta_{j_{k-1}}$  second, and so on. This allows us to apply Lemma 1 repeatedly, giving us

$$\begin{aligned} \text{wt}(\Delta_J u) &= t^{2 \text{des}(u)} q^{2 \text{maj}(u)} \frac{\prod_{j \in J \setminus \text{Des}(u)} tq^j}{\prod_{i \in J \cap \text{Des}(u)} tq^i} \\ &= t^{\text{des}(u)} q^{\text{maj}(u)} \prod_{j \in J \Delta \text{Des}(u)} tq^j, \end{aligned}$$

as desired. □

Now we can prove Theorem 6.

*Proof of Theorem 6.* By Lemma 2, it suffices to show that

$$B'(u) = \{\Delta_J u : J \subset [n]\}.$$

Because the two sets have the same cardinality, it suffices to show  $\Delta_J u \in B'(u)$  for any  $J$ . This is clear since  $|\Delta_J u| = u$ . □

## 5. THE BIJECTION

From Lemma 2, we notice in particular that there is a unique signed permutation with the same flag statistics as  $u$  has ordinary statistics, obtained by taking  $J = \text{Des}(u)$ . Define

$$w_{\min}(u) = \Delta_{\text{Des}(u)} u.$$

Given any signed permutation  $w$  we have  $\Delta_i^2 w = w$  and, if  $i < j$ ,

$$\Delta_i \Delta_j w = \Delta_j \Delta_i w = w_1 \cdots w_i \overline{w_{i+1}} \cdots \overline{w_j} w_{j+1} \cdots w_n.$$

In particular the composite operator  $\Delta_i \Delta_{i+1}$  simply negates letter  $i+1$ . Since the  $\Delta_i$  commute and generate all possible sign changes, we can identify permutations in  $B'(u)$  with subsets of  $[n]$  by the correspondence

$$J \longleftrightarrow \Delta_J w_{\min}(u) = \Delta_{J \Delta \text{Des}(u)} u.$$

Lemma 2 can now be interpreted to show that the boolean algebra generated by subsets of  $[n]$  respects the statistics (fdes, fmaj) on  $B'(u)$ . Specifically, if  $w \in B'(u)$  corresponds to  $J = \{j_1, \dots, j_k\}$ , then  $\text{fdes}(w) = \text{des}(u) + k$  and  $\text{fmaj}(w) = \text{maj}(u) + (j_1 + \cdots + j_k)$ .

See Figure 2 for an example with  $u = 3142$ .

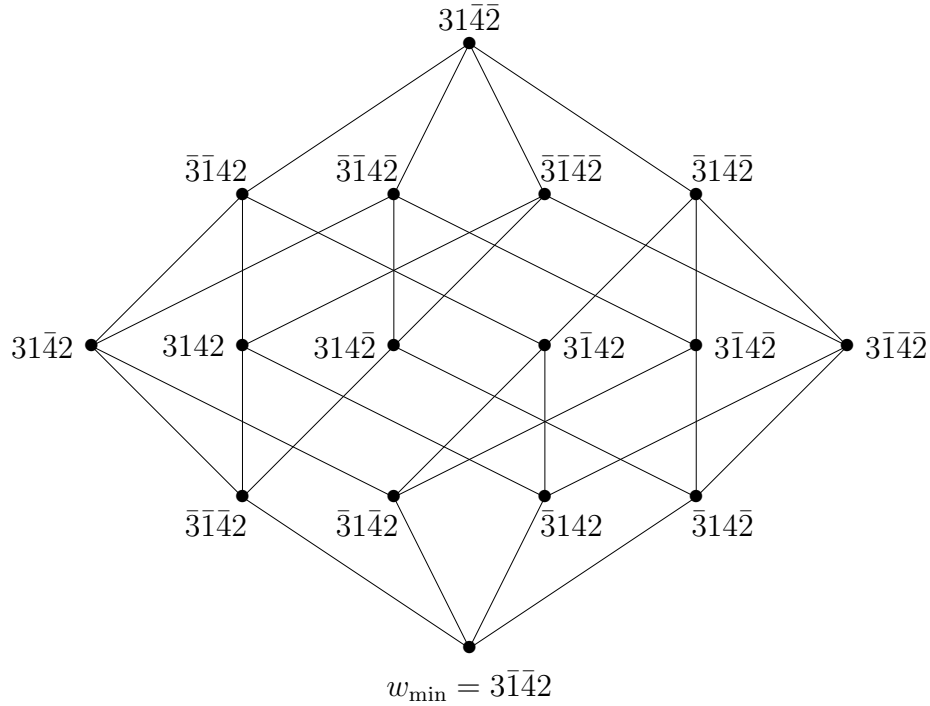


FIGURE 2.  $B'(u)$  for  $u = 3142$

Implicit in here is the bijection that Adin, Brenti, and Roichman wanted.

**Theorem 7.** *The bijection*

$$u_J \longleftrightarrow \Delta_{J \Delta \text{Des}(u)} u$$

*is weight-preserving in that*

$$\text{ndes}(u_J) = \text{fdes}(\Delta_{J \Delta \text{Des}(u)} u)$$

$$\text{nmaj}(u_J) = \text{fmaj}(\Delta_{J \Delta \text{Des}(u)} u).$$

**Example.** Let  $w = \bar{2}1\bar{3}\bar{4}$  with  $\text{ndes}(w) = 4$  and  $\text{nmaj}(w) = 11$ . We first find  $u = \text{st}(w) = 1423$ . Since  $J = \{2, 3, 4\}$  is the set of negative numbers in  $w$  and  $\text{Des}(u) = \{2\}$ , we take  $v = \Delta_3\Delta_4(1423) = 142\bar{3}$ , with the desired flag descent and flag major numbers,  $\text{fdes}(v) = 4$  and  $\text{fmaj}(v) = 11$ .

In the other direction, let  $v = 43\bar{1}\bar{2}$  with  $\text{fdes}(v) = 4$  and  $\text{fmaj}(v) = 8$ . First we see that  $u = |v| = 4312$  and  $v = \Delta_2\Delta_4u$ . Since  $\text{Des}(u) = \{1, 2\}$ , we get  $J = \{1, 4\}$  and  $u_J = 32\bar{1}\bar{4}$  with  $\text{ndes}(u_J) = 4$  and  $\text{nmaj}(u_J) = 8$ , as desired.

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